Auto-Bäcklund Transformation and Solitary-wave Solutions to Non-integrable Generalized Fifth-order Nonlinear Evolution Equations

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We show that the application of the truncated Painlevé expansion and symbolic computation leads to a new class of analytical solitary-wave solutions to the general fifth-order nonlinear evolution equations which include Lax, Sawada-Kotera (SK), Kaup-Kupershmidt (KK), and Ito equations. Some explicit solitary-wave solutions are presented.

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While it is easy to write down in closed from a solitary wave solution for the simplest standard model, namely the Korteweg-de Vries (KdV) equation, it has proved quite difficult to obtain such solutions for the problems from which the KdV equation was derived as a First approximation [1]. As such KdV hierarchy models, we investigate the generalized non-integrable fifth-order nonlinear evolution equations of the form [2]

$$u_t + \alpha u u_{xxx} + \beta u_x u_{xx} + \gamma u^2 u_x + u_{xxxxx} = 0, \quad (1)$$

where α, β, and γ are constant model parameters. This model includes the Lax [3], Kaup-Kupershmidt (KK) [4 - 7], Sawada-Kotela (SK) [9], and Ito equations [8]. As the constants α, β, and γ take different values, the properties of (1) drastically change. For instance, the Lax equation with $\alpha = 10, \beta = 20$, and $\gamma = 30$, and the SK equation where $\alpha = \beta = \gamma = 5$, are completely integrable. These two equations have N-soliton solutions and an infinite set of conservation laws. The KK equation, with $\alpha = 10, \beta = 25$, and $\gamma = 20$, is also known to be integrable [6] and to have bilinear representations [10, 11], but the explicit form of its N-soliton solution is apparently not known. A fourth equation in the model is the Ito equation, with $\alpha = 3, \beta = 6$, and $\gamma = 2$, which is not completely integrable but has a limited number of conservation laws [8]. More recently, using a simplified version of Hirota's method, Hereman and Nuseir [2] explicitly constructed multi-soliton solutions of the KK equation for which soliton solutions were not previously known. However, to our knowledge no attempt has been made for finding more general solitary-wave solutions other than the above mentioned models because of lengthy and nearly impossible calculations without proper symbolic computation packages. By utilizing the symbolic package *Maple*, Hong [12] recently found some analytical solitary-wave solutions to the general fifth-order water models in [13], with some constraints on the model parameters.

In this work, we make use of both the truncated Painlevé expansion and the symbolic computation method [14 - 17] to obtain an *auto-Bäcklund transformation* and certain explicit solitary-wave solutions for the generalized non-integrable fifth-order nonlinear evolution equation, which are different from the solutions of the models mentioned above [3 - 8].

A non-linear partial differential equation (NPDE) is said to possess the Painlevé property when the solutions of the NPDE are "single valued" about the movable, singularity manifold which is "non-characteristic". To be more precise, if the singularity manifold is determined by

$$\phi(z_1, z_2, ..., z_n) = 0, \tag{2}$$

and $u = u(z_1, z_2, ..., z_n)$ is a solution of the NPD, then it is required that

$$u = \phi^{\alpha} \sum_{j=0}^{\infty} u_j \phi^j, \tag{3}$$

where $u_0 \neq 0, \phi = \phi(z_1, z_2, ..., z_n), u_j = u_j(z_1, z_2, ..., z_n)$ are analytical function of (z_j) in the neighborhood of the manifold (3), and α is a negative, rational number. Substitution of (3) into the NPDE determines the allowed values of α and defines the recursion relation for u_j, $j = 0, 1, 2, ...$. When the ansatz (3) is correct, the NPDE is said to possess the Painlevé property and is conjectured to be integrable [14, 17].

In order to find solitonic solutions for (1), we truncate the Painlevé expansion, (3), at the constant-level term in the senses of Tian and Gao [15] and Khater et al. [17],

$$u(x,t) = \phi^{-J}(x,t) \sum_{l=0}^{J} u_l(x,t)\phi^l(x,t). \tag{4}$$

On balancing the highest-order contributions from the linear term (i. e. u_{xxxxx}) with the highest order contributions from the nonlinear term (i.e, $u^2 u_x$), we get $J = 2$, so that

$$u(x,t) = \frac{u_0(x,t)}{\phi^2(x,t)} + \frac{u_1(x,t)}{\phi(x,t)} + u_2(x,t). \tag{5}$$

We will stay with the general assumption that $\phi_x \neq 0$ but will not initially impose any constraints on the model parameters α, β, γ. When substituting the above expressions into (1) with the symbolic program package *Maple*, we let the coefficients of like powers of ϕ vanish, so as to get the set of Painlevé-Bäcklund (PB) equations,

$$\phi^{-7}: \; -720\, u_0 {\phi_x}^5 - 12\,\beta\, {u_0}^2 {\phi_x}^3 - 24\,\alpha\, {u_0}^2 {\phi_x}^3 - 2\,\gamma\, {u_0}^3 \phi_x = 0, \tag{6}$$

$$\begin{aligned}\phi^{-6}: \; & -30\,\alpha\, u_1 u_0 {\phi_x}^3 + 18\,\alpha\, {u_0}^2 \phi_x \phi_{xx} + 18\,\alpha\, u_0 {\phi_x}^2 u_{0,x} + 1200\, u_0 {\phi_x}^3 \phi_{xx} - 10\,\beta\, u_0 {\phi_x}^3 u_1 - 4\,\beta\, {u_0}^2 \phi_x \phi_{xx} \\ & + 14\,\beta\, u_0 {\phi_x}^2 u_{0,x} - 120\, u_1 {\phi_x}^5 - 5\,\gamma\, {u_0}^2 u_1 \phi_x + \gamma\, {u_0}^2 u_{0,x} + 600\, {\phi_x}^4 u_{0,x} = 0,\end{aligned} \tag{7}$$

$$\begin{aligned}\phi^{-5}: \; & -6\,\alpha\, u_0 \phi_x u_{0,xx} - 4\,\gamma\, {u_0}^2 \phi_x u_2 - 24\,\alpha\, u_2 u_0 {\phi_x}^3 + 6\,\beta\, u_1 {\phi_x}^2 u_{0,x} - 2\,\beta\, {u_1}^2 {\phi_x}^3 + 4\,\beta\, u_0 \phi_x u_1 \phi_{xx} \\ & + 18\,\alpha\, u_1 {\phi_x}^2 u_{0,x} + 24\,\alpha\, u_1 u_0 \phi_x \phi_{xx} + 2\,\gamma\, u_0 u_1 u_{0,x} - 4\,\gamma\, u_0 {u_1}^2 \phi_x + 6\,\alpha\, u_0 {\phi_x}^2 u_{1,x} + 10\,\beta\, u_{1,x} u_0 {\phi_x}^2 \\ & - 2\,\beta\, u_0 \phi_x u_{0,xx} - 6\,\alpha\, u_0 \phi_{xx} u_{0,x} - 2\,\beta\, u_{0,x} u_0 \phi_{xx} - 4\,\beta\, \phi_x {u_{0,x}}^2 - 720\, {\phi_x}^2 \phi_{xx} u_{0,x} + 120\, {\phi_x}^4 u_{1,x} \\ & + 240\, u_1 {\phi_x}^3 \phi_{xx} + \gamma\, {u_0}^2 u_{1,x} - 240\, u_0 {\phi_x}^2 \phi_{xxx} - 360\, u_0 \phi_x {\phi_{xx}}^2 - 2\,\alpha\, {u_0}^2 \phi_{xxx} - 6\,\alpha\, {u_1}^2 {\phi_x}^3 \\ & - 240\, {\phi_x}^3 u_{0,xx} = 0,\end{aligned} \tag{8}$$

$$\begin{aligned}\phi^{-4}: \; & \gamma\, {u_1}^2 u_{0,x} - \gamma\, {u_1}^3 \phi_x + 90\, {\phi_{xx}}^2 u_{0,x} + 60\, {\phi_x}^2 u_{0,xxx} + \beta\, u_{0,x} u_{0,xx} - 6\,\alpha\, u_2 u_1 {\phi_x}^3 + 18\,\alpha\, u_2 u_0 \phi_x \phi_{xx} \\ & + 2\,\gamma\, u_0 u_2 u_{0,x} - 3\,\alpha\, u_0 u_1 \phi_{xxx} + 2\,\gamma\, u_0 u_1 u_{1,x} + 6\,\alpha\, u_1 {\phi_x}^2 u_{1,x} + 4\,\beta\, u_1 {\phi_x}^2 u_{1,x} + 18\,\alpha\, u_2 {\phi_x}^2 u_{0,x} \\ & + 6\,\alpha\, u_1 \phi_x u_{0,xx} - \beta\, u_1 \phi_x u_{0,xx} - \beta\, u_{0,x} u_1 \phi_{xx} - 6\,\alpha\, u_1 \phi_{xx} u_{0,x} + 6\,\alpha\, {u_1}^2 \phi_x \phi_{xx} + \beta\, {u_1}^2 \phi_x \phi_{xx} \\ & - 6\,\gamma\, u_1 u_2 u_0 \phi_x - 2\,\beta\, u_0 \phi_x u_{1,xx} + 6\,\beta\, u_{2,x} u_0 {\phi_x}^2 - 2\,\beta\, u_{1,x} u_0 \phi_{xx} - 3\,\alpha\, u_0 u_{1,x} \phi_{xx} - 3\,\alpha\, u_0 u_{1,xx} \phi_x \\ & + 180\, \phi_x \phi_{xx} u_{0,xx} - 6\,\beta\, u_{0,x} u_{1,x} \phi_x - 60\, {\phi_x}^3 u_{1,xx} + 120\, \phi_x \phi_{xxx} u_{0,x} + \gamma\, {u_0}^2 u_{2,x} - 60\, u_0 \phi_{xx} \phi_{xxx} \\ & + \alpha\, u_0 u_{0,xxx} - 180\, {\phi_x}^2 u_{1,x} \phi_{xx} - 60\, u_1 {\phi_x}^2 \phi_{xxx} - 90\, u_1 \phi_x {\phi_{xx}}^2 + 30\, u_0 \phi_x \phi_{xxx} = 0,\end{aligned} \tag{9}$$

$$\phi^{-3}: \ \alpha u_0 u_{1,xxx} + \gamma u_{1,x} {u_1}^2 - 2\beta u_{2,x} u_0 \phi_{xx} - 2\beta u_0 \phi_x u_{2,xx} + 2\gamma u_0 u_1 u_{2,x} - 2 u_0 \phi_t - 2 u_0 \phi_{xxxxx}$$
$$- 4\beta u_{2,x}\phi_x u_{0,x} + 20 u_{1,xxx}{\phi_x}^2 + 30 {\phi_{xx}}^2 u_{1,x} - 20\phi_{xx} u_{0,xxx} - 10\phi_x u_{0,xxxx} - 10\phi_{xxxx} u_{0,x}$$
$$- 20\phi_{xxx} u_{0,xx} + 6\alpha u_2 u_1 \phi_x \phi_{xx} + 2\gamma u_1 u_2 u_{0,x} + 2\beta u_{2,x} u_1 {\phi_x}^2 - 3\alpha u_1 u_{1,xx}\phi_x - 3\alpha u_1 u_{1,x}\phi_{xx}$$
$$- \beta u_{1,x} u_1 \phi_{xx} - \beta u_{1,xx} u_1 \phi_x + 2\gamma u_{1,x} u_0 u_2 - 2\alpha u_2 u_0 \phi_{xxx} - 2\gamma {u_2}^2 u_0 \phi_x + 6\alpha u_2 {\phi_x}^2 u_{1,x}$$
$$- 6\alpha u_2 \phi_x u_{0,xx} - 6\alpha u_2 \phi_{xx} u_{0,x} - 2\gamma {u_1}^2 u_2 \phi_x + 40\phi_x u_{1,x}\phi_{xxx} + \alpha u_1 u_{0,xxx} - 2\beta {u_{1,x}}^2 \phi_x$$
$$+ \beta u_{1,xx} u_{0,x} + \beta u_{1,x} u_{0,xx} + 60\phi_x u_{1,xx}\phi_{xx} + 10 u_1 \phi_x \phi_{xxxx} + 20 u_1 \phi_{xx}\phi_{xxx} - \alpha {u_1}^2 \phi_{xxx}$$
$$+ 10\alpha {\phi_x}^2 u_{1,xxx} - 3\mu\phi_x u_{0,xx} - 5\alpha\phi_x u_{0,xxxx} - 4\gamma {u_{1,x}}^2 \phi_x + 3\mu {\phi_x}^2 u_{1,x} = 0, \quad (10)$$

$$\phi^{-2}: \ -u_1\phi_t + \alpha u_2 u_{0,xxx} - u_1 \phi_{xxxxx} - 2\beta u_{2,x} u_{1,x}\phi_x + 2\gamma u_1 u_2 u_{1,x} - \gamma {u_2}^2 u_1 \phi_x - \alpha u_2 u_1 \phi_{xxx}$$
$$+ 2\gamma u_{2,x} u_0 u_2 - 3\alpha u_2 u_{1,x}\phi_{xx} - 3\alpha u_2 u_{1,xx}\phi_x - \beta u_{2,xx} u_1 \phi_x - \beta u_{2,x} u_1 \phi_{xx} + u_{0,xxxxx} + u_{0,t}$$
$$+ \beta u_{1,x} u_{1,xx} - 10 u_{1,xx}\phi_{xxx} - 5 u_{1,x}\phi_{xxxx} + \gamma {u_2}^2 u_{0,x} + \gamma u_{2,x} {u_1}^2 + \alpha u_1 u_{1,xxx} - 5 u_{1,xxxx}\phi_x$$
$$+ \alpha u_0 u_{2,xxx} - 10 u_{1,xxx}\phi_{xx} + \beta u_{2,x} u_{0,xx} + \beta u_{2,xx} u_{0,x} = 0, \quad (11)$$

$$\phi^{-1}: \ \alpha u_1 u_{2,xxx} + \alpha u_2 u_{1,xxx} + u_{1,xxxxx} + 2\gamma u_1 u_2 u_{2,x} + \gamma {u_2}^2 u_{1,x} + \beta u_{1,x} u_{2,xx} + \beta u_{2,x} u_{1,xx} + u_{1,t} = 0, \quad (12)$$

ϕ^0 : u_2 needs to satisfy the original equation, i.e.,

$$u_{2,xxxxx} + \beta u_{2,x} u_{2,xx} + \alpha u_2 u_{2,xxx} + u_{2,t} + \gamma {u_2}^2 u_{2,x} = 0 \quad (13)$$

The set of (5) and (6 - 13) constitutes an *auto-Bäcklund transformation*, if the set is solvable with respect to $\phi(x,t), u_0(x,t), u_1(x,t)$ and $u_2(x,t)$ [15, 16]. Equation (6) brings out three possibilities for $u_0(x,t)$:

$$u_0^{\mathrm{I}} = 0, \quad u_0^{\mathrm{II}} = 3\,\frac{(-\beta - 2\alpha + \mathcal{H})\,{\phi_x}^2}{\gamma}, \quad u_0^{\mathrm{III}} = 3\,\frac{(-\beta - 2\alpha - \mathcal{H})\,{\phi_x}^2}{\gamma}, \quad (14)$$

where $\mathcal{H} \equiv \sqrt{\beta^2 + 4\beta\alpha + 4\alpha^2 - 40\gamma}$. Some complicated symbolic manipulations are required to find general solutions for the remaining $u_1(x,t)$ and $u_2(x,t)$. However, due to increasing complexities in the symbolic computations, a long CPU time is required. Thus, in the following, in order to shorten the computation time, we constrain the model parameters by requiring

$$\mathcal{H} = 0 \Longrightarrow \alpha = -\beta/2 + \sqrt{10\gamma} \quad \text{or} \quad \alpha = -\beta/2 - \sqrt{10\gamma}. \quad (15)$$

In the rest of the paper, we consider a solution family for the case of the non-trivial solution u_0^{II} with the first constraint $\alpha = -\beta/2 + \sqrt{10\gamma}$. Of course, other classes of solitary-wave solutions can be found from u^{III} and the second constraint on α, β, γ.

After substituting u_0^{II} and α into (7), we obtain

$$u_1(x,t) = \frac{12\phi_{xx}(5\beta + 2\sqrt{10\gamma})}{\beta\sqrt{10\gamma} + 4\gamma}. \quad (16)$$

Subsequently, we get the following solution for $u_2(x,t)$ from (9):

$$u_2(x,t) = \frac{-32\gamma\phi_x\phi_{xxx} + 15\beta^2{\phi_{xx}}^2 - 20\beta^2\phi_x\phi_{xxx} - 4\phi_x\phi_{xxx}\sqrt{10\gamma}\beta + 24{\phi_{xx}}^2\gamma}{{\phi_x}^2(\sqrt{10\gamma}\beta^2 + 8\gamma\beta + 16\gamma^{3/2})}. \quad (17)$$

(a) (b)

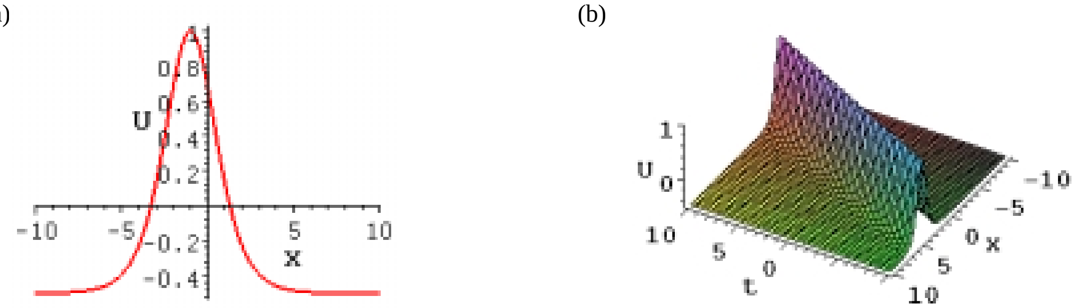

Fig. 1. (a) Typical sech^2-shpaed solitary-wave solution $u(x,0)$ in (21) with parameters $A = 1, B = 1, C_1 = 1, \beta = -2$, $\gamma = 10$, and $\alpha = -\beta/2 + \sqrt{10\gamma} = 11$. (b) $u(x,t)$ shows the solitary wave property that the amplitude becomes finite as $|x|$ approaches infinity.

Thus, we are able to find a class of analytical solutions $u(x,t)$ in terms of $u_0^{\mathrm{II}}, u_1, u_2$ in (14 - 17) with an arbitrary function $\phi(x,t)$ constrained by the remaining equations (10 - 13). We note that, once a Bäcklund transformation is discovered and a set of "seed" solutions is given, one will be able to find an infinite number of solutions by repeated applications of the transformation, i. e., to generate a hierarchy of solutions with increasing complexity. In the rest of the paper we will find a family of some exact analytical solutions of (1).

Sample Solution

A trial solution

$$\phi(x,t) = 1+e^{\mathcal{Q}(x,t)}, \text{ where } \mathcal{Q}(x,t) = A(t)x+B(t) \quad (18)$$

is substituted into the remaining constraint equations (10 - 13). From (10), we find

$$\phi^{-3}\&\ x : A(t)' = 0 \ \Rightarrow\ A(t) = A = \text{constant}. \quad (19)$$

Then, we obtain $B(t)'$ from the terms of ϕ^{-3} and integrate it over t to get

$$B(t) = \frac{-\mathcal{R}\cdot t - \mathcal{S}}{4\mathcal{T}},$$

$$\begin{aligned}\mathcal{R} \equiv\ & 10500\,A^5\sqrt{10}\beta^6\gamma - 62720\,A^5\sqrt{10}\gamma^3\beta^2 \\ &+ 13750\,A^5\beta^7\sqrt{\gamma} - 97280\,A^5\gamma^{7/2}\beta \\ &- 201600\,A^5\beta^3\gamma^{5/2} + 14000\,A^5\beta^5\gamma^{3/2} \\ &- 28000\,A^5\sqrt{10}\beta^4\gamma^2 - 6144\,A^5\sqrt{10}\gamma^4 \\ &+ 625\,A^5\beta^8\sqrt{10},\end{aligned}$$

$$\begin{aligned}\mathcal{S} \equiv\ & -224000\,C_1\beta^3\gamma^{5/2} - 2500\,C_1\beta^7\sqrt{\gamma} \\ &- 84000\,C_1\beta^5\gamma^{3/2} - 71680\,C_1\gamma^{7/2}\beta \\ &- 53760\,C_1\sqrt{10}\gamma^3\beta^2 - 4096\,C_1\sqrt{10}\gamma^4 \\ &- 7000\,C_1\sqrt{10}\beta^6\gamma - 56000\,C_1\sqrt{10}\beta^4\gamma^2,\end{aligned}$$

$$\begin{aligned}\mathcal{T} \equiv\ & 1024\gamma^4\sqrt{10} + 1750\sqrt{10}\beta^6\gamma + 14000\sqrt{10}\beta^4\gamma^2 \\ &+ 13440\sqrt{10}\gamma^3\beta^2 + 56000\,\beta^3\gamma^{5/2} + 625\,\beta^7\sqrt{\gamma} \\ &+ 21000\,\beta^5\gamma^{3/2} + 17920\,\gamma^{7/2}\beta, \quad (20)\end{aligned}$$

where C_1 is the constant of integration. Combining all terms, we find a family of analytical solutions of (1) as

$$\begin{aligned}u(x,t) =\ & 3\,\frac{(-\beta - 2\,\alpha + \mathcal{H})\,{\phi_x}^2}{\gamma}\cdot\frac{1}{(1+e^{\mathcal{Q}(x,t)})^2} \\ &+ \frac{12\phi_{xx}(5\,\beta + 2\sqrt{10\gamma})}{\beta\sqrt{10\gamma} + 4\,\gamma}\cdot\frac{1}{1+e^{\mathcal{Q}(x,t)}} \\ &- \Big[32\,\gamma\,\phi_x\phi_{xxx} - 15\,\beta^2{\phi_{xx}}^2 + 20\,\beta^2\phi_x\phi_{xxx} \\ &\quad + 4\,\phi_x\phi_{xxx}\sqrt{10\gamma}\beta - 24\,{\phi_{xx}}^2\gamma\Big] \\ &\Big/\Big[{\phi_x}^2(\sqrt{10\gamma}\beta^2 + 8\,\gamma\,\beta + 16\gamma^{3/2})\Big] \quad (21)\end{aligned}$$

with the trial function $\phi(x,t)$ in (18) with $\mathcal{Q}(x,t)$ in (19, 20).

In the following we show that our solutions indeed have solitary-wave properties by presenting some figures from the family: We choose, as an example, a set of arbitrary constants; $A = 1, B = 1$, and $C_1 = 1$

(a) (b)

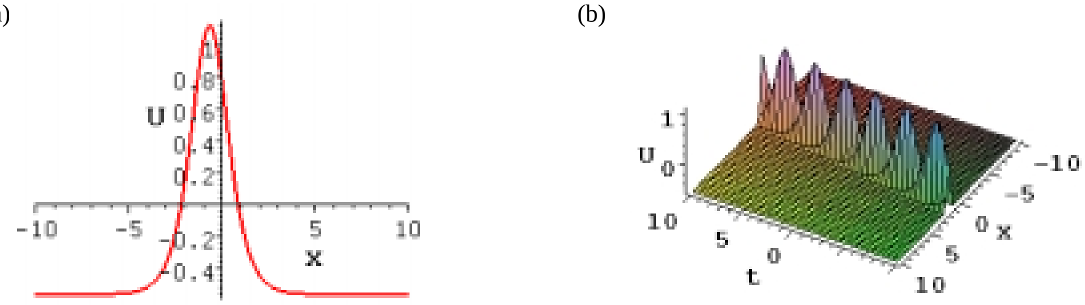

Fig. 2. (a) Sech^2-shaped solitary-wave solution $u(x,0)$ with parameters $A = 1, B = 1, C_1 = 1, \beta = 2, \gamma = 40$, and $\alpha = 19$. (b) $u(x,t)$ shows the breather solitary-wave solution behavior.

with the constraint $\alpha = -\beta/2 + \sqrt{10\gamma}$. Figures 1 and 2 correspond to ($\beta = -2, \gamma = 10, \alpha = 11$) and ($\beta = 2, \gamma = 40, \alpha = 19$), respectively. Firstly, we note that $u(x,0)$ in Figs. 1(a) and 2(a) are both sech$(x)^2$-shaped solutions. From Figs. 1(b) and 2(b) we understand that in both cases the solutions have solitary wave property, i. e., $u(x,t)$ tends to a finite value as $|x|$ approaches infinity. Interestingly, Fig. 2(b) shows a breather solitary-wave solution.

To sum up, with symbolic computations and the truncated Painlevé expansion analysis, we showed that Bäcklund ransformations exist for the generalized fifth-order non-integrable nonlinear evolution equation. We found a class of analytical solutions $u(x,t)$ to (1) in terms of $u_0^{\rm II}, u_1, u_2$ in (14 - 17) with an arbitrary function $\phi(x,t)$ constrained by (10 - 13). A sample solution family for $u(x,t)$ was found in Eq.(21) with the constraint $\alpha = -\beta/2 + \sqrt{10\gamma}$. More solitary-wave families can be found from $u^{\rm III}(x,t)$ and other constraints on α, β, γ in (15).

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[1] S. Kichenassamy, Nonlinearity **10**, 133 (1997).
[2] W. Hereman and A. Nuseir, Mathematics and Computers in Simulation, **Vol. 43**, 13 (1997).
[3] P. Lax, Comm. Pure Appl. Math. **21**, 467 (1986).
[4] A. Fordy and J. Gibbons, Phys. Lett. **75A**, 325 (1980).
[5] R. Hirota and A. Ramani, Phys. Lett. **76A**, 95 (1980).
[6] D. Kaup, Stud. Appl. Math. **62**, 189 (1980).
[7] B. A. Kupershmidt, Phys. Lett. **102A**, 213 (1984).
[8] M. Ito, J. Phys. Soc. Japan **49**, 771 (1980).
[9] P. J. Caudrey and J. D. Gibbon, Proc. Roy. Soc. London **A 358**, 287 (1977).
[10] M. Jimbo and T. Miwa, Publ. RIMS, Kyoto University **19**, 943(1983).
[11] J. Satsuma and D. J. Kaup, J. Phys. Soc. Japan **43**, 692 (1977).
[12] W. Hong, Il. Nuovo Cim. **B**, in press (1999).
[13] S. Kichenassamy and P. J. Olver, SIAM J. Math. Anal. **23**, 1141 (1992).
[14] J. Weiss, M. Tabor, and G. Carnevale, J. Math. Phys. **24**, 522 (1983).
[15] B. Tian and Y. T. Gao, Phys. Lett **209A**,(1995) 297.
[16] W. P. Hong and Y. D. Jung, Phys. Lett. **257A**, 149 (1999).
[17] A. H. Khater, D. K. Callebaut, A. B. Shamardan, and R. S. Ibrahim, Phys. of Plasmas, **Vol. 5**, 395 (1998).